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ON CONVECTIVE MOTIONS OF VISCOPLASTIC FLUID IN A POROUS MEDIUM*

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The two-dimensional free convection of viscoplastic fluid in a porous medium heated from the side is investigated, and the problem of convection in an infinite vertical layer is solved. The variational principle which makes possible the determination of the threshold Rayleigh number is formulated in the case of an arbitrary plane region. The problem of convection in a rectangular region is solved using the net-point method.

1. Consider the convective motion of a viscoplastic fluid saturating a closed region of a porous medium heated from the side. The equations of viscoplastic fluid convection in a porous medium in the Boussinesq approximation in dimensionless form are

$$\nabla p - RT\gamma = \mathbf{F}, \quad R = \frac{\rho_f g \beta \Theta k a}{\eta\chi} \tag{1.1}$$

$$\partial T/\partial t + \mathbf{v}\nabla T = \Delta T, \quad \operatorname{div} \mathbf{v} = 0$$

$$\mathbf{v} = -\left(1 - \frac{\Lambda}{|\mathbf{F}|}\right)\mathbf{F}\,(|\mathbf{F}| > \Lambda), \quad \mathbf{v} = 0\,(|\mathbf{F}| \leqslant \Lambda); \quad \Lambda = \frac{\lambda k a}{\eta\chi} \tag{1.2}$$

The impermeability condition $(\mathbf{vn}) = 0$, where $\mathbf{n}$ is a normal to the surface bounding that region, and specify a temperature distribution corresponding to heating from the side.

As units of length, time, velocity, pressure, and temperature we select, respectively, a, $(\rho c_p)_m a^2/(\rho c_p)_f\chi$, χ/a, $\eta\chi/k$, $\eta\chi/ka$, Θ (a is a characteristic dimension of the region, Θ is a characteristic temperature difference, η is the dynamic viscosity of the fluid, and k is the permeability coefficient).

Here p is the contribution of convection to pressure, T is the temperature measured from some mean value, ρ_f is the fluid density, β is the coefficient of thermal expansion of the fluid, g is the acceleration of gravity, γ is a unit vector directed vertically upward, $\chi = \varkappa_m/(\rho c_p)_f$ is the thermal diffusivity coefficient, $\varkappa_m$ is the coefficient of thermal conductivity of the medium, and c_p is the specific heat. Quantities denoted by subscript f pertain to the fluid, those with subscript m to the porous medium saturated with fluid.

Equations (1.2) define the relation between the filtration rate $\mathbf{v}$ and the averaged force $\mathbf{F}$ of interaction between the fluid and structure of the medium (the law of filtration with limit gradient /1/).

The problem contains two dimensionless parameters: the Rayleigh number R and the rheological parameter Λ.

2. The simplest flow arises in a long vertical layer whose boundaries are maintained at different temperatures. A similar problem was solved in the case of homogeneous fluid in /2/, where it was shown that the motion is of the form of two convection counterflows with a core flow zone formed in the central parts of both streams. It will be shown below that in the case of a porous medium a single stagnation zone is present in the layer central part.

Consider an infinite vertical layer whose boundaries $x = \mp 1/2$ are maintained at constant but different temperatures $T = \mp 1/2$. If we assume that the velocity has only a vertical component $v_y = v$ and the temperature depends only on x, we obtain from the heat conduction equation $T = x$. It is possible to show in the usual way (see /3/) that $\partial p/\partial y = C = \text{const}$. Then using the y component of the equation of motion we obtain

$$v = Rx - C - \Lambda, \ v > 0; \ v = Rx - C + \Lambda, \ v < 0 \tag{2.1}$$

in which the velocity distribution satisfies the condition of the stream closure only for $C = 0$.

It follows from (2.1) that the stagnation zone boundaries are defined by the formula

$$x = \pm \Lambda/R \tag{2.2}$$

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For $R < 2\Lambda$ the stagnation zone occupies the total cross section of the channel and convection motion is absent. In the region of $R > 2\Lambda$ the convection intensity increases as R increases. The flow rate through half-cross section of the channel is $Q = (R - 2\Lambda)^2/(8R)$.

Thus convective motion in a vertical layer of porous medium exists only at Rayleigh numbers higher than some threshold value

$$R_* = 2\Lambda \tag{2.3}$$

with motion intensity increasing with R according to a quadratic law near the threshold, and to a linear law at distance from it.

3. Consider the more general case of convective motion of viscoplastic fluid filling an arbitrary plane region D bounded by contour Γ_e in a porous medium. As in the case of the plane vertical layer, the convection motion with heating from the side is only possible at Rayleigh numbers exceeding a certain threshold R_*. We formulate the variational principle for the determination of R_*. Let the closed contour Γ lie entirely inside the region. We integrate along Γ the equation of motion

$$\oint_\Gamma \mathbf{F} d\mathbf{\Gamma} + R \oint_\Gamma \gamma T d\mathbf{\Gamma} = 0 \tag{3.1}$$

where it is taken into account that pressure is a single-valued function of coordinates. Since $|\mathbf{F}| < \Lambda$ for $R < R_*$, for the first integral in (3.1) we have

$$\left| \oint_\Gamma \mathbf{F} d\mathbf{\Gamma} \right| \leqslant \Lambda L \tag{3.2}$$

where L is the contour length. Using the Stokes theorem we transform the second of integrals (3.1) in the integral over area S bounded by contour Γ, and obtain

$$R\Sigma \leqslant \Lambda L, \quad \Sigma = \left| \int_S \frac{\partial T}{\partial x} ds \right| \tag{3.3}$$

where the x-coordinate is horizontal.

Let us show that the region in which at $R > R_*$ there exists some closed contour Γ_0 as $R \to R_*$.

Assume the opposite, i.e. that at $R = R_*$ we have throughout the region $\mathbf{F} = \Lambda \mathbf{k}$, where $\mathbf{k}$ is a unit vector. We apply to the equation of motion the operation rot and project the obtained equation on the z-axis. Then

$$\Lambda \left(\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} \right) + R = 0 \tag{3.4}$$

We denote by φ the angle between vector $\mathbf{k}$ and the x-axis, and write Eq.(3.4) in the form

$$\Lambda \mathbf{k} \nabla \varphi + R = 0 \tag{3.5}$$

whose integration along some closed streamline yields

$$\Lambda \oint \frac{\partial \varphi}{\partial l} dl + RL = 0$$

where L is the contour length.

For force $\mathbf{F}$ to be a single-valued function of coordinates it is necessary that

$$\oint \frac{\partial \varphi}{\partial l} dl = 2\pi n \quad (n = 0, 1, 2, \ldots) \tag{3.6}$$

This implies that $L = 2\pi (\Lambda/R)\, n$, i.e. we have a discrete set of lengths L. If the flow region width is finite, length L varies continuously from one streamline to another and cannot be discrete. Consequently, the flow cannot begin at once throughout the region, but is generated at $R = R_*$ along the thin closed contour.

Since the velocity vector is directed along the tangent to Γ_0 as $R \to R_*$, the equality in (3.3) is attained only at $R = R_*$ with Γ coinciding with Γ_0. This yields the following variational principle for R_*:

$$\frac{R_*}{\Lambda} = \min_\Gamma \Phi, \quad \Phi = \frac{L}{\Sigma} \tag{3.7}$$

where Φ is the functional of the form of contour Γ. The contour Γ which yields the minimum functional Φ is the same as contour Γ_0.

Note that the variational principle (3.7) is closely related to the variational principle established in /1/.

At high Rayleigh numbers close to R_* it is, indeed, possible to disregard the temperature field distortion due to convection, as well as the term containing the square of velocity. Then the variational principle assumes the form /1/

$$\int_D (\Lambda |\mathbf{v}| - RT\gamma\mathbf{v})\, ds = \min \tag{3.8}$$

Since motion is impossible at $R < R_*$, the functional can have only nonnegative values, with its zero value at $\mathbf{v} = 0$. This implies that

$$R_* = \min \{\int |\mathbf{v}|\, ds \,|\int T\gamma\mathbf{v} ds|^{-1}\} \tag{3.9}$$

Bearing in mind that we have a plane problem we introduce the stream function $\psi(v_x = \partial\psi/\partial y, v_y = -\partial\psi/\partial x)$ and represent the integral in the denominator in the form

$$\int T\gamma\mathbf{v}\, ds = \int \frac{\partial T}{\partial x} \psi\, ds \tag{3.10}$$

Assume that the motion begins on the closed contour Γ. It is convenient to pass in the variational principle (3.9) from variation with respect to $\mathbf{v}$ to varying the form of contour Γ. Inside that contour $\psi = q = \text{const}$, where q is the flow rate through the contour cross section. Since velocity is directed along the contour, the integral in the numerator can be written as

$$\int |\mathbf{v}|\, ds = qL \tag{3.11}$$

where L is the contour length. Substituting (3.10) and (3.11) into (3.9), we obtain a variational principle that coincides with (3.7).

Investigations are particulary simple in the case when the heating conditions (in the region of $R < R_*$) correspond to a constant temperature gradient. Then $\partial T/\partial x = \text{const} = A$, and it is possible to set $A = 1$ without loss of generality. Then obviously $\Sigma = S$, and the variational principle assumes the simple form

$$R_* = \Lambda \min_{\Gamma} \frac{L}{S} \tag{3.12}$$

Such extension of contour Γ decreases functional Φ, hence contour Γ_0 at least touches the boundary of region D, and individual sections of Γ_0 and Γ_e may even coincide. We denote the sections of contour Γ_0 that wholly lie in region D by Γ_0', and those at the boundary of D by Γ_0''. It can be shown that all Γ_0' have no discontinuities. Since two-sided variations are possible in these sections, Euler equations of the form

$$\left(\frac{y'}{\sqrt{1+y'^2}}\right)' + \Phi = 0 \tag{3.13}$$

must be satisfied in these sections. It is assumed that on a given section contour Γ_0' is defined by the expression $y = y(x)$. The general solution of Eq. (3.13) is of the form

$$(x + C_1)^2 + (y + C_2)^2 = 1/\Phi^2 \tag{3.14}$$

where C_1 and C_2 are constants of integration that may differ in various Γ_0'. When considering the variation of Γ in a region adjacent to any junction point of Γ_0' and Γ_0'' it is possible to show that such junction must be smooth.

Thus contour Γ_0 consists of separate sections of the boundary of region D and circular arcs of radius $r = 1/\Phi$ inscribed in boundary D.

Let us consider in detail the case of a rectangular region.

$$x \in [-1/2,\ 1/2],\ y \in [-l/2,\ l/2] \tag{3.15}$$

where the rectangle base is taken as the unit of length and the y-axis is vertical. For a fixed r it is possible to effect in this case the indicated separation in a unique manner, after which functional Φ becomes a function of r. Finally r is obtained either by using the self-consistency condition $\Phi_m = 1/r$ or the condition of minimum $d\Phi/dr = 0$. The same result is evidently obtained in both cases for the threshold value of the Rayleigh number

$$R_* = \Phi_m \Lambda,\ \Phi_m = 1/r = [l + 1 + \sqrt{(l-1)^2 + \pi l}]/l \tag{3.16}$$

We have the following values of Φ_m for several specific cases. In a square region ($l = 1$) $\Phi_m \approx 3.77$ and when $l = 5$, $\Phi_m \approx 2.58$. At the limit as $l \to \infty$ we have $\Phi_m \to 2$ and $R_* = 2\Lambda$, which conforms with formula (2.3) obtained above for the vertical layer.

4. Let us now pass to the numerical investigation of the finite-amplitude convective motion of viscoplastic fluid in a porous medium. We shall consider convective motion in the rectangular region (3.15). Conditions of impermeability, and for flow rate and temperature distribution at the region boundary appropriate for heating from the side are specified at the region boundaries.

Model (1.2) is not suitable for numerical investigation, since it is then necessary to seek separate solutions for stagnation and viscoplastic flow zones, with merging of solutions to be carried out at the a priori unknown zone interfaces. Because of this, we use the regularized model

$$\mathbf{F} = -\left(1 + \frac{\Lambda}{\varepsilon + |\mathbf{v}|}\right)\mathbf{v} \tag{4.1}$$

which, strictly speaking, does not allow for stagnation zones. However, for fairly small regularization parameter ε formula (4.1) conforms closely to the law of filtration with a limit gradient. In applying formula (4.1) stagnation zones can only be considered as indicating that the rate of filtration in such zones is low in comparison with the rate in the remaining part of the region. As the criterion isolating stagnation zones we can take

$$|\mathbf{F}| < \Lambda \tag{4.2}$$

We write the equations of convection in terms of stream function ψ and temperature T as

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} + R\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial \psi}{\partial x} = \Delta T \tag{4.3}$$

$$F_x = -\zeta\frac{\partial \psi}{\partial y}, \quad F_y = \zeta\frac{\partial \psi}{\partial x}, \quad \zeta = 1 + \Lambda\left[\varepsilon + \sqrt{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2}\right]^{-1}$$

with boundary conditions

$$\psi|_{\Gamma_e} = 0, \quad T\left(-\frac{1}{2}, y\right) = -\frac{1}{2}, \quad T\left(\frac{1}{2}, y\right) = \frac{1}{2}, \quad T\left(x, -\frac{l}{2}\right) = T\left(x, \frac{l}{2}\right) = x \tag{4.4}$$

Problem (4.3), (4.4) was solved by the method of finite differences. Steady solutions were obtained using the method of establishment with explicit finite-difference scheme. All three-dimensional derivatives appearing in equations were approximated by central differences.

Computations were carried out for $l = 1$ and $l = 5$ and rheological parameters (Λ, ε): (2.5; 0.1), (2.5; 0.05). The Rayleigh number was varied in the interval 0 to 200. Basic computations were carried out on a uniform net of pitch 1/15 for $l = 5$ and 1/20 for $l = 1$.

Besides the stream function and temperature fields we determined in the course of computations the integral characteristics of the flow: extremal value of the stream function ψ_m and the dimensionless heat flux N

$$N = \frac{1}{2l}\oint_{\Gamma_e}\left|\frac{\partial T}{\partial n}\right| dl \tag{4.5}$$

(the dimensionless stream was normalized so that $N = 1$ corresponded to the heat-conducting mode). Obtained numerical dependence of the dimensionless heat flux on the Rayleigh number are shown in Fig.1 for rheological parameters $\Lambda = 2, 5$, with $\varepsilon = 0.1$ for the rectangular $l = 5$ and square $l = 1$ regions (curves *1* and *2*, respectively). The threshold values of Rayleigh numbers R_{1*} and R_{2*} obtained using formula (3.16) are indicated on the R-axis for $l = 5$ and $l = 1$, respectively. For comparison, curves of $N(R)$ function for a Newtonian fluid $(\Lambda = 0)$ for $l = 5$ and $l = 1$ are shown for comparison in Fig.1 by dash lines *1* and *2*, respectively. A sharp increase of convection intensity can be seen in the region of $R \sim R_*$. The weak convection at $R < R_*$ is linked with the use of the regularized model (4.1) instead of that with the limit gradient (1.2). At high Rayleigh numbers the motion acquires the character of the boundary layer flow, as is also the case of the Newtonian flow (*). A closed boundary layer is formed with a core of comparatively low mobility and a vertical temperature gradient. The dependence of heat flux on the Rayleigh number becomes exponential.

*) M.P. Vlasiuk and V.I. Polezhaev, Natural convection and heat transfer in permeable porous materials. Preprint, No.77, Inst. Problem Mekhan., Akad..Nauk.SSSR, 1975.

The disposition of stagnation zones for $l=5$ and $R=10$, 20, 50 is shown in Figs.2,a,b and c, respectively (stagnation zones are shaded). Computations have shown that at low Rayleigh numbers the stagnation zone extends to the whole region. At $R\sim R_*$ a narrow zone of viscoplastic flow adjacent along its separate sections to the cavity walls is generated (Fig.2,a). The form of contour Γ_0 calculated by the variational principle (3.12) is shown by the dash curve in the same figure. It can be seen that the forms of contour Γ_0 and of the viscoplastic flow zone numerically calculated for $R\gtrsim R_*$ are reasonably close. They differ in that the boundary of the corner stagnation zone calculated numerically are convex relative to the center, which is, apparently, due to the use of the regularized filtration model.

With increasing Rayleigh number the central stagnation zone gradually diminishes (Figs, 2,b,c).

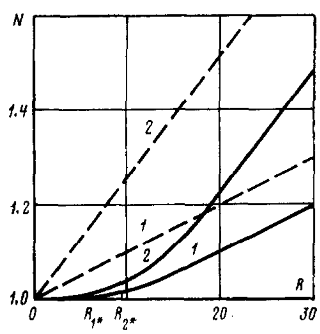

Fig.1

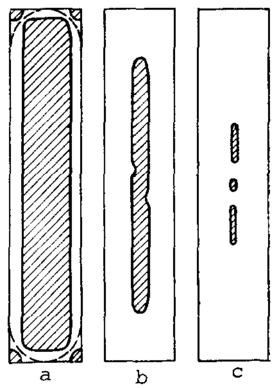

Fig.2

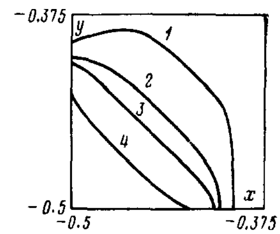

Fig.3

To elucidate the effect of the form of the boundary of the corner stagnation zone in a net cell and of the regularization parameter ε on the form of boundary computations were carried out for fixed values of the geometric parameter $l=1$, Rayleigh number $R=12$, and the rheological parameter $\Lambda=2.5$, with $h=1/40$, $\varepsilon=0.025\div10^{-5}$. Boundaries of the corner stagnation zone are shown in Fig.3 for $\varepsilon=0.025$, 0.025/2, 0.025/4, 0.025/32 by curves *1–4*, respectively. It will be seen that when the regularization parameter is decreased, i.e. when approaching conditions of the filtration law with limit gradient, the boundary of the corner stagnation zone convex relative to the region center becomes convex to the stagnation zone, which is in agreement with the results obtained with the use of the variational principle.

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REFERENCES

1. BERNADINER M.G. and ENTOV V.M., Hydrodynamic Theory of Filtration of Unusual Fluids. Moscow, NAUKA, 1975.

2. YANG WEN-JET and YEH HSU-CHIEN, Free convection flow of Bingham plastic between two vertical plates. Trans. ASME, Ser. C.J. Heat Transfer, Vol.87, No.2, 1965.

3. GERSHUNI G.Z. and ZHUKHOVITSKII E.M., Convection Stability of Incompressible Fluid. Moscow, NAUKA, 1972.

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